Some properties of minimizers for the L^1TV functional

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October 3, 2006

Abstract

We present two results characterizing minimizers of the L^1TV functional $F(u) \equiv \int |\nabla u| dx + \lambda \int |u - f| dx$; $u, f : \mathbb{R}^n \to \mathbb{R}$. If we restrict to $u = \chi_\Sigma$ and $f = \chi_\Omega$, $\Sigma, \Omega \in \mathbb{R}^n$, the L^1TV functional reduces to $E(\Sigma) = \text{Per}(\Sigma) + \lambda|\Sigma \triangle \Omega|$. We show that there is a minimizer Σ such that its boundary $\partial\Sigma$ lies between the union of all balls of radius $\frac{n}{\lambda}$ contained in Ω and the corresponding union of $\frac{n}{\lambda}$-balls in Ω^c. We also show that if a ball of radius $\frac{n}{\lambda} + \epsilon$ is almost contained in Ω, a slightly smaller concentric ball can be added to Σ to get another minimizer. Finally, we comment on recent results Allard has obtained on L^1TV minimizers and how these relate to our results.

1 Introduction

Much of the work in image analysis reduces to extracting and processing information from images. Much of that information is, in turn, carried by shapes present in the images. The methods for extracting information from images range broadly over stochastic, wavelet, and variational or PDE based methods. In the past five to ten years, the variational and related PDE methods have drawn a great deal of attention.

In this paper we study one of these variational methods from a shape processing perspective. More specifically, we establish new results concerning the properties of exact minimizers for the rather new L^1TV functional. While this functional is applicable to scalar functions on $\mathbb{R}^n$, we study the functional specialized to binary functions, i.e. binary images or shapes.

The minimization of the L^1TV functional,

$$u^* = \text{argmin} \int |\nabla u| dx + \lambda \int |u - f| dx, \tag{1}$$

yields *denoised images* u that are smoothed yet close, in an L^1 sense, to the *measures image* f (sometimes called the *input image* or *noisy measurement*). As is well known from studies of the Rudin-Osher-Fatemi total variation functional [7],

$$u^* = \text{argmin} \int |\nabla u| dx + \lambda \int |u - f|^2 dx, \tag{2}$$

the total variation term $\int |\nabla u| dx$ reduces oscillations while permitting sharp edges, something that previous methods could not do or did very poorly.

The change of the *data fidelity term* ($\int |u - f|^2 dx$ in (2)) to the L^1 term in the L^1TV functional has the effect of making that functional much more natural from a geometric point of view.

The L^1TV functional was studied very carefully in a paper by Chan and Esedoglu [3]. (The discrete analog of the L^1TV functional had been previously studied by Alliney [2] and Nikolova [6].) Chan and Esedoglu [3] show that for binary input images, there are also binary minimizers. More precisely, given any minimizer to (1) with binary input, almost every super-levelset is the support of a binary minimizer of the same functional. For binary input χ_Ω, the functional can therefore be written as

$$\Sigma^* = \operatorname{argmin} E(\Sigma) \equiv \operatorname{Per}(\Sigma) + \lambda |\Sigma \bigtriangleup \Omega|, \tag{3}$$

where Σ, Σ^* and Ω are the supports of the binary functions under study. Allard has recently submitted a paper [1] in which he uses very intricate geometric measure theory techniques to prove precise regularity results for minimizers of a class of functionals which includes the L^1TV functional. We comment a bit more on Allard's work in the final section of the paper.

Our results for minimizers of the L^1TV functional can be viewed results on the regularization of noisy shapes. The first result gives us a characterization of minimizers for the case in which the noise expresses itself as perturbations of the boundary. The second result characterizes the L^1TV regularization of a binary images with measurement noise. In discrete images this corresponds to pixels flipping from 0 to 1 or 1 to 0 as driven by the noise process.

Now a brief outline of the paper. In the next section we present the results for the case of Σ, Σ^* and Ω in $\mathbb{R}^2$. This is the case most relevant for typical images. In the Section 3 we prepare for the proof of these results by introducing, in some detail, the notion of measure theoretic boundary, exterior and interior. This permits us to avoid the intricacies of the notion of reduced boundary. Next we prove the results for sets in $\mathbb{R}^2$ (Section 4). This section is the longest and most involved. In Section 5, we state the theorems for the case $n > 2$ noting a few modifications that must be made. Since all the hard parts of the proof for $n > 2$ are contained in the $n = 2$ case, we do not present the proof details. We close (Section 6) with a brief discussion of our results and their relation to one of Allard's results.

In what follows we represent minimizers of the L^1TV functional (3) by Σ, dropping the superscript $*$ used above.

2 Main Results (n = 2)

The two main results of this paper can be stated informally as follows. Define $R \equiv 2/\lambda$. For any $\epsilon_1, \epsilon_2 > 0$,

(1) any ball of radius R completely contained in Ω is also contained in Σ, and

(2) if a ball of radius $R + \epsilon_1$ is almost contained in Ω, then a concentric ball of radius $R - \epsilon_2$ is completely contained in Σ.

More precisely we have,

Theorem 1. *Let Ω be a bounded, measurable subset of $\mathbf{R}^2$. Let Σ be any solution of (3). Assume that a ball B_R of radius R lies completely in Ω: $B_R \subset \Omega$. Then $B_R \cup \Sigma$ is also a minimizer. Moreover, if $B_R \subset \Omega^c$, then $(B_R \cup \Sigma^c)^c$ is also a minimizer.*

and,

Theorem 2. *Given $\hat{r} \in (R, \frac{\sqrt{7}}{2}R)$ and $\epsilon \in (0, 1 - \frac{1}{\sqrt{2}})$, we can choose $\delta = \delta(R, \hat{r}, \epsilon) > 0$ such that*

$$|B_{\hat{r}} \setminus \Omega| < \delta \Rightarrow B_{(1-\epsilon)R} \subset \Sigma. \tag{4}$$

Remark 1. *Obvious analogs of these theorems hold in $\mathbb{R}^n$ with modifications commented on in Section 5.*

Remark 2. *Theorem 1 and the lower semicontinuity of the L1TV functional implies that there is a minimizer Σ such that $\bigcup\{B_{\frac{2}{\lambda}}(x) \subset \Omega\} \subset \Sigma$ and $\bigcup\{B_{\frac{2}{\lambda}}(x) \subset \Omega^c\} \subset \Sigma^c$.*

Remark 3. *These theorems are close to optimal since the minimizer for $\Omega = B_{\frac{2}{\lambda}-\eta}$ for arbitrarily small $\eta > 0$ has unique minimizer $\Sigma = \emptyset$.*

3 Measure Theoretic Boundary

To simplify our analysis of the energy $E(\Sigma) \equiv \mathrm{Per}(\Sigma) + \lambda|\Omega \bigtriangleup \Sigma|$, we introduce measure theoretic boundary, interior, and exterior.

Define $\mathrm{Per}(\Sigma) \equiv \int |\nabla \chi_\Sigma| dx$. We say a set in $\mathbb{R}^n$ is a *set of finite perimeter* if $\mathrm{Per}(\Sigma) < \infty$. The structure theorem for sets of finite perimeter tells us that $\mathrm{Per}(\Sigma) = H^{n-1}(\partial^*\Sigma)$, where $\partial^*\Sigma$ is the *reduced boundary* of Σ. The reduced boundary is rather complicated to define and difficult to manipulate. Instead, we use another theorem which asserts $\partial^*\Sigma \subset \partial_*\Sigma$ and $H^{n-1}(\partial_*\Sigma - \partial^*\Sigma) = 0$ to conclude that $\mathrm{Per}(\Sigma) = H^{n-1}(\partial_*\Sigma)$, where $\partial_*\Sigma$ denotes the *measure theoretic boundary* of Σ. (See [5] Theorem 2, Section 5.7 and Lemma 1, Section 5.8 for more details.) We now define measure theoretic boundary, interior, and exterior.

Definition 1. *A point $x \in \mathbb{R}^n$ is in $\partial_* A$, the* measure theoretic boundary *of A if*

$$\limsup_{r \to 0} \frac{\mathcal{L}(B(x,r) \cap A)}{r^n} > 0 \tag{5}$$

and

$$\limsup_{r \to 0} \frac{\mathcal{L}(B(x,r) \cap A^c)}{r^n} > 0. \tag{6}$$

A point $x \in \mathbb{R}^n$ *is in* A^i_*, *the* measure theoretic interior *of* A *if*

$$\limsup_{r \to 0} \frac{\mathcal{L}(B(x,r) \cap A^c)}{r^n} = 0. \tag{7}$$

while $x \in \mathbb{R}^n$ *is in* A^o_*, *the* measure theoretic exterior *of* A *if*

$$\limsup_{r \to 0} \frac{\mathcal{L}(B(x,r) \cap A)}{r^n} = 0 \tag{8}$$

Lemma 1. *Let* A *be a subset of* $\mathbb{R}^n$ *with finite perimeter. Then*

1. $Per(A) = H^{n-1}(\partial_* A)$
2. $R^n = (A^o_*) \cup (\partial_* A) \cup (A^i_*)$ *and the three sets are pairwise disjoint.*

Proof: (1) As stated above this follows from [5] Theorem 2, Section 5.7 and Lemma 1, Section 5.8. (2) This follows directly from Definition 1. ■

Lemma 2. *Suppose* A *and* B *be subsets of* $\mathbb{R}^n$. *Then:*

1. *if* $x \in A^i_*$ *or* $x \in B^i_*$ *then* $x \in (A \cup B)^i_*$.
2. $\partial_*(A \cup B) \subset \partial_* A \cup \partial_* B$
3. $\partial_* A \cup \partial_* B = (\partial_* A \cap B^i_*) \cup (\partial_* A \cap B^o_*) \cup (\partial_* B \cap A^i_*) \cup (\partial_* B \cap A^o_*) \cup (\partial_* A \cap \partial_* B)$
4. *(1-3) immediately imply that* $\partial_*(A \cup B) \subset (\partial_* A \cap B^o_*) \cup (\partial_* B \cap A^o_*) \cup (\partial_* A \cap \partial_* B)$
5. $(\partial_* A \cap B^o_*) \cup (\partial_* B \cap A^o_*) \subset \partial_*(A \cup B)$
6. $\partial_* A^c = \partial_* A$.
7. $(A^c)^o_* = A^i_*$.

Proof: The lemma follows in a straightforward manner from the definitions of measure theoretic boundary, interior and exterior.■

Corollary 1. *If* $H^{n-1}(\partial_* A \cap \partial_* B) = 0$ *then*

1. $H^{n-1}(\partial_*(A \cup B)) = H^{n-1}(\partial_* A \cap B^o_*) + H^{n-1}(\partial_* B \cap A^o_*)$
2. $H^{n-1}(\partial_*(A \cap B)) = H^{n-1}(\partial_* A \cap B^i_*) + H^{n-1}(\partial_* B \cap A^i_*)$

Proof: (1): Lemma 2:(4-5) imply that

$$\begin{aligned} H^{n-1}(\partial_* A \cap B^o_*) \; &+ \; H^{n-1}(\partial_* B \cap A^o_*) && (9)\\ &\le \; H^{n-1}(\partial_*(A \cup B)) && (10)\\ &\le \; H^{n-1}(\partial_* A \cap B^o_*) + H^{n-1}(\partial_* B \cap A^o_*) + H^{n-1}(\partial_* A \cap \partial_* B) && (11)\end{aligned}$$

and the conclusion follows. (2): This follows from (1), Lemma 2:(6)-(7) and the fact that $A \cap B = (A^c \cup B^c)^c$.
■

Remark 4. *Since $\partial_* A = (\partial_* A \cap B^i_*) \cup (\partial_* A \cap \partial_* B) \cup (\partial_* A \cap B^o_*)$, the assumption that $H^{n-1}(\partial_* A \cap \partial_* B) = 0$ means we can, for the sake of measurement, consider $\partial_* A = (\partial_* A \cap B^i_*) \cup (\partial_* A \cap B^o_*)$.*

Remark 5. *Now suppose that $H^{n-1}(\partial_* A) < \infty$ and B_r is the ball of radius r centered at $x \in \mathbb{R}^n$ (we suppress the x). Note that $\partial_* B_r = \partial B_r$. By the coarea formula, the set of r's such that $H^{n-1}(\partial_* B_r \cap \partial_* A) > 0$ is at most countable. We conclude that the r's for which $H^{n-1}(\partial_* B_r \cap \partial_* A) = 0$ are dense and have full measure in $\mathbb{R}$.* For the rest of this section we assume that we have chosen r such that $H^{n-1}(\partial_* A \cap \partial_* B_r) = 0$.

Theorem 3. *Suppose $B_r \subset \Omega$. Define $E(\Sigma) \equiv \int |\nabla \chi_\Sigma| dx + \lambda \int |\xi_\Sigma - \xi_\Omega| dx = Per(\Sigma) + \lambda |\Sigma \triangle \Omega|$. Then*

$$\Delta E = E(\Sigma \cup B_r) - E(\Sigma) = -H^{n-1}(\partial_* \Sigma \cap (B_r)^i_*) + H^{n-1}(\partial_* B_r \cap \Sigma^o_*) - \lambda |B_r \setminus \Sigma|.$$

Proof: Since $\mathrm{Per}(\Sigma \cup B_r) = H^{n-1}(\partial_* \Sigma \cap (B_r)^o_*) + H^{n-1}(\partial_* B_r \cap \Sigma^o_*)$ and $\mathrm{Per}(\Sigma) = H^{n-1}(\partial_* \Sigma \cap (B_r)^i_*) + H^{n-1}(\partial_* \Sigma \cap (B_r)^o_*)$ we get $\mathrm{Per}(\Sigma \cup (B_r)) - \mathrm{Per}(\Sigma) = -H^{n-1}(\partial_* \Sigma \cap (B_r)^i_*) + H^{n-1}(\partial_* B_r \cap \Sigma^o_*)$. Noting that $B_r \subset \Omega$ implies $|(\Sigma \cup B_r) \triangle \Omega| - |\Sigma \triangle \Omega| = |B_r \setminus \Sigma|$ finishes the proof. ■

The example A's and B's in Figure 1 illustrate why the above care is necessary.

4 The Comparisons: proofs of Theorems 1 and 2

Recall that Σ denotes a minimizer of 3. If $\Delta E \equiv E(B_r \cup \Sigma) - E(\Sigma) \le 0$ then $B_r \cup \Sigma$ must also be a minimizer.

Proof of Theorem 1: Computing ΔE for $B_r \subset \Omega$ we get (for all but countably many r):

$$\begin{aligned} \Delta E = &\; -H^1(\partial_* \Sigma \cap {B_r}^i_*) + H^1(\partial_* B_r \cap \Sigma^o_*) - \lambda |B_r \setminus \Sigma| && (12)\\ = &\; -H^1(\partial_* \Sigma \cap {B_r}^i_*) - H^1(\partial_* B_r \cap \Sigma^i_*) + H^1(\partial_* B_r \cap \Sigma^o_*) + H^1(\partial_* B_r \cap \Sigma^i_*) \\ &\; -\lambda |B_r \setminus \Sigma| - \lambda |B_r \cap \Sigma| + \lambda |B_r \cap \Sigma| && (13)\\ = &\; -H^1(\partial_*(B_r \cap \Sigma)) + H^1(\partial_*(B_r)) - \lambda |B_r| + \lambda |B_r \cap \Sigma| && (14)\\ = &\; \left(H^1(\partial_*(B_r)) - \lambda |B_r|\right) + \left(\lambda |B_r \cap \Sigma| - H^1(\partial_*(B_r \cap \Sigma))\right). && (15)\end{aligned}$$

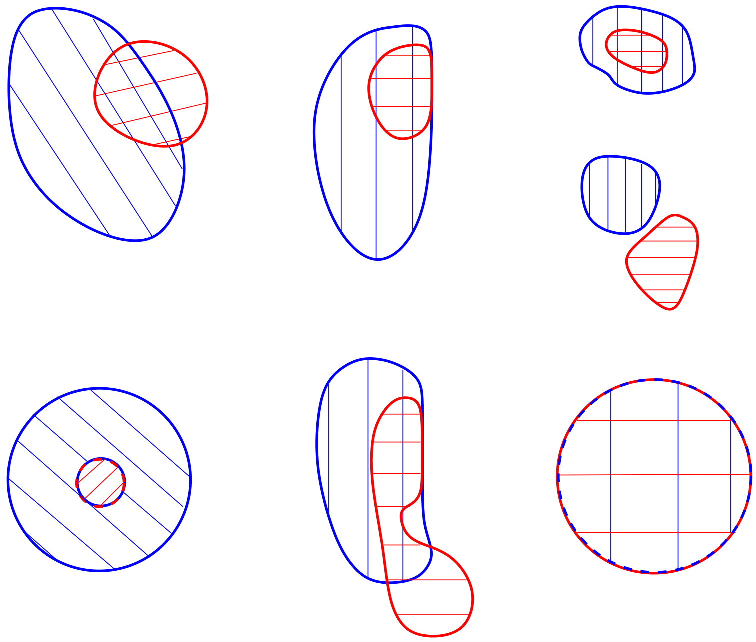

Figure 1: Illustration of the cases one needs to consider in order to understand how $\partial_*(A \cup B)$ relates to $\partial_* A$ and $\partial_* B$

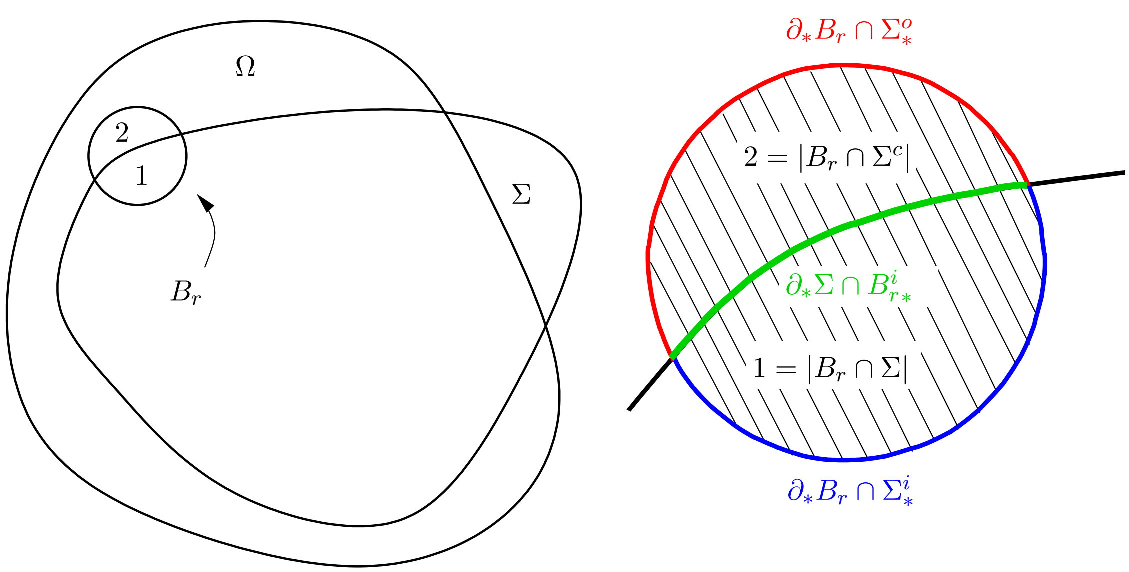

Figure 2: An illustration useful for computing ΔE when $B_r \cap \Omega^c = \emptyset$.

This is illustrated in Figure 2 below.

Now we choose a sequence of radii $r_i < R$ converging to R, for which (15) holds.

Defining ρ_i by $\pi\rho_i^2 = |B_{r_i} \cap \Sigma|$, ρ_i^* by $2\pi\rho_i^* = H^1(\partial_*(B_{r_i} \cap \Sigma))$ and remembering that $R \equiv \frac{2}{\lambda}$, we have that

$$\Delta E = \left(2\pi r_i - \tfrac{2}{R}\pi r_i^2\right) + \left(\tfrac{2}{R}\pi\rho_i^2 - 2\pi\rho_i^*\right) \tag{16}$$

$$= 2\pi r_i(1 - \tfrac{r_i}{R}) + 2\pi\rho_i(\tfrac{\rho_i}{R} - \tfrac{\rho_i^*}{\rho_i}). \tag{17}$$

Note that the isoperimetric inequality gives $\frac{\rho_i^*}{\rho_i} \geq 1$ for all i, that $\frac{\rho_i}{R} < 1$ for all i and that $(1 - \frac{r_i}{R}) \to_{i\to\infty} 0$. The right hand side of (17) converges therefore to zero. Using the fact that ΔE is lower semicontinuous for sequences in L^1, (which follows from the lower semicontinuity of the BV seminorm), we conclude that $\Delta E(B_R) \leq 0$. We conclude that $\Sigma \cup B_R$ is also a minimizer.

Finally, we note that $E_\Omega(\Sigma) \equiv \mathrm{Per}(\Sigma) + \lambda|\Sigma \vartriangle \Omega| = E_{\Omega^c}(\Sigma^c) \equiv \mathrm{Per}(\Sigma^c) + \lambda|\Sigma^c \vartriangle \Omega^c|$. From this we deduce that Σ minimizes $E_\Omega \Leftrightarrow \Sigma^c$ minimizes E_{Ω^c}. Therefore, $B_R \subset \Omega^c$ implies $(\Sigma^c \cup B_R)^c$ is also a minimizer. ■

Proof of Theorem 2: In the case that $B_r \cap \Omega^c \neq \emptyset$,

$$\begin{aligned}
\Delta E &= -H^1(\partial_*\Sigma \cap B_{r*}^{\ i}) + H^1(\partial_* B_r \cap \Sigma_*^o)) - \lambda|B_r \setminus \Sigma| + 2\lambda|B_r \cap \Omega^c \cap \Sigma^c| && (18)\\
&\leq -H^1(\partial_*\Sigma \cap B_{r*}^{\ i}) + H^1(\partial_* B_r \cap \Sigma_*^o) - \lambda|B_r \setminus \Sigma| + 2\lambda|B_r \cap \Omega^c| && (19)\\
&= -H^1(\partial_*\Sigma \cap B_{r*}^{\ i}) + H^1(\partial_* B_r \cap \Sigma_*^o) - \lambda|B_r \setminus \Sigma| + 2\lambda|B_r \setminus \Omega| && (20)\\
&= \left(H^1(\partial_*(B_r)) - \lambda|B_r|\right) + \left(\lambda|B_r \cap \Sigma| - H^1(\partial_*(B_r \cap \Sigma))\right) + 2\lambda|B_r \setminus \Omega| && (21)\\
&= (\mathrm{Per}(B_r) - \lambda|B_r|) + (\lambda|B_r \cap \Sigma| - \mathrm{Per}(B_r \cap \Sigma)) + 2\lambda|B_r \setminus \Omega|. && (22)
\end{aligned}$$

This is illustrated in Figure 3 below.

Lemma 3. $|B_R \setminus \Sigma| \leq 6\delta$

Proof: Since we assume Σ is a minimizer, $\Delta E \geq 0$. We will perturb with balls of radius $r \leq \hat{r}$. Then, $|B_r \setminus \Omega| \leq |B_{\hat{r}} \setminus \Omega| := \delta$. These assumptions together with (22) and the isoperimetric inequality $(\mathrm{Per}(B_r \cap \Sigma) \geq 2\sqrt{\pi}|B_r \cap \Sigma|^{\frac{1}{2}})$ imply:

$$\begin{aligned}
0 &\leq \Delta E \leq 2\pi r - \lambda\pi r^2 + \lambda|B_r \cap \Sigma| - 2\sqrt{\pi}|B_r \cap \Sigma|^{\frac{1}{2}} + 2\lambda\delta. && (23)\\
&= \lambda|B_r \cap \Sigma| - 2\sqrt{\pi}|B_r \cap \Sigma|^{\frac{1}{2}} + \Big(2\lambda\delta + 2\pi r - \lambda\pi r^2\Big) && (24)\\
&= f(\xi) \equiv \lambda\xi^2 - 2\sqrt{\pi}\xi + \Big(2\lambda\delta + 2\pi r - \lambda\pi r^2\Big),\ (\xi \equiv |B_r \cap \Sigma|^{\frac{1}{2}}) && (25)\\
&= \frac{2}{R}\xi^2 - 2\sqrt{\pi}\xi + \Big(\frac{4\delta}{R} + 2\pi r - \frac{2\pi r^2}{R}\Big)\ (\text{recalling } R = \frac{2}{\lambda})\ . && (26)
\end{aligned}$$

In view of the last inequality, we describe values of ξ for which $f(\xi) \geq 0$. For a given r, the zeros of $f(\xi)$ are at:

$$\xi_\pm(r) = \sqrt{\frac{\pi R^2}{4}} \ \pm\ \sqrt{\frac{\pi R^2}{4} + (\pi r(r - R)) - 2\delta}. \tag{27}$$

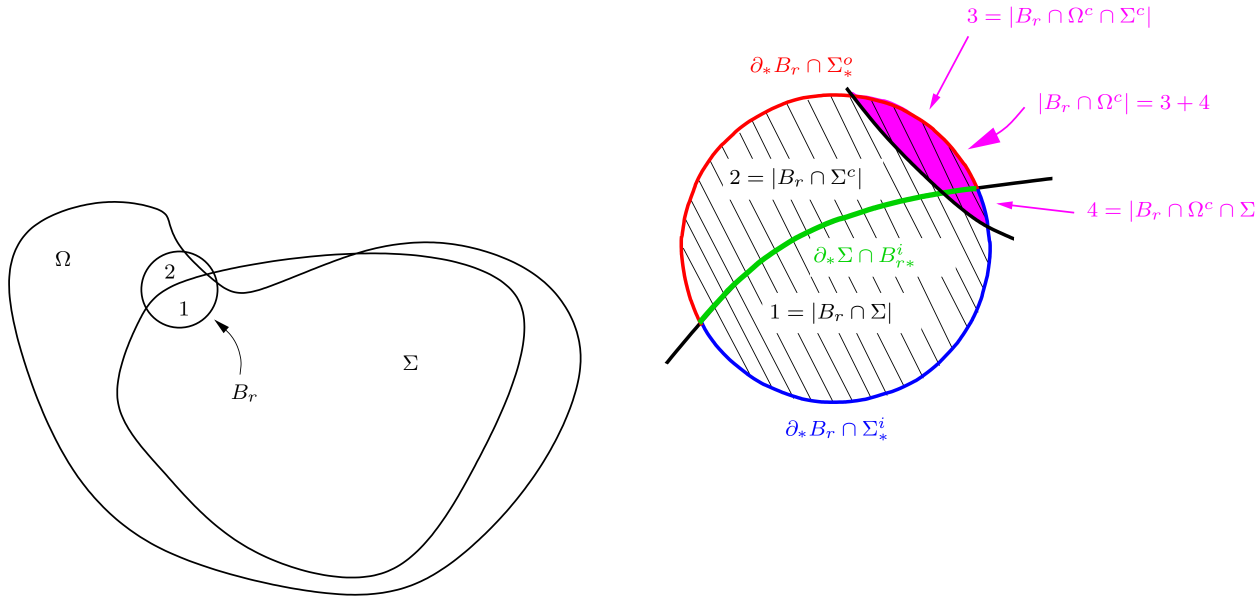

Figure 3: An illustration useful for computing ΔE when $B_r \cap \Omega^c \neq \emptyset$

Thus, for all $r \leq \hat{r}$, we have:

$$\text{Either } |B_r \cap \Sigma|^{\frac{1}{2}} \leq \xi_-(r), \text{ or } |B_r \cap \Sigma|^{\frac{1}{2}} \geq \xi_+(r). \tag{28}$$

If we take $r = \hat{r} > R$, then $2\pi r(r - R) > 0$, and assuming

Condition 1.

$$\delta < \frac{\pi \hat{r}}{2}(\hat{r} - R) \tag{29}$$

implies $\xi_-(r) < 0$. This implies

$$|B_{\hat{r}} \cap \Sigma| \geq \xi_+^2(\hat{r}) > \pi R^2. \tag{30}$$

Since $r = \hat{r} < \frac{\sqrt{7}}{2} R$ we get

$$|B_R \cap \Sigma| > |B_r \cap \Sigma| - \frac{3\pi R^2}{4} > \frac{\pi R^2}{4} \tag{31}$$

Now we consider $\xi_\pm(R)$:

$$\xi_\pm(R) = \sqrt{\frac{\pi R^2}{4}} \pm \sqrt{\frac{\pi R^2}{4} - 2\delta}. \tag{32}$$

Assuming

Condition 2.

$$\delta < \frac{\pi R^2}{8}, \tag{33}$$

we get that $\xi_{\pm}(R)$ are real and distinct. Since

$$\xi_-^2(R) < \frac{\pi R^2}{4} < |B_R \cap \Sigma|, \tag{34}$$

we conclude that

$$\xi_+^2 \leq |B_R \cap \Sigma|. \tag{35}$$

Computing, we get

$$\begin{aligned}
\xi_+^2 &= \left(\sqrt{\frac{\pi R^2}{4}} + \sqrt{\frac{\pi R^2}{4} - 2\delta}\right)^2 && (36)\\
&= \frac{\pi R^2}{2} - 2\delta + \frac{\pi R^2}{2}\sqrt{1 - \frac{8\delta}{\pi R^2}} && (37)\\
&\geq \frac{\pi R^2}{2} - 2\delta + \frac{\pi R^2}{2}\left(1 - \alpha\frac{8\delta}{\pi R^2}\right) \quad \left(\text{assuming } \delta \leq \frac{\pi R^2}{8}\frac{2\alpha - 1}{\alpha^2}\right) && (38)\\
&= \pi R^2 - (2 + 4\alpha)\delta. && (39)
\end{aligned}$$

Choosing $\alpha = 1$, and noting that Condition 2 then implies the assumption in 38 is satisfied, we get

$$|B_R \cap \Sigma| \geq \xi_+^2 \geq \pi R^2 - 6\delta. \tag{40}$$

This gives

$$|B_R \setminus \Sigma| \leq 6\delta \tag{41}$$

as advertised. ■

Remark 6. *What if either $\hat{r}$ or R are radii such that (22) (and therefore (27)) does not hold? We can simply choose another $\tilde{r} < \hat{r}$ arbitrarily close to $\hat{r}$, for which (22) does hold. The $\tilde{\delta} \equiv |B_{\tilde{r}} \setminus \Omega|$ will be no greater than, and arbitrarily close to, δ. As we will see, the only conditions on δ that are not functions of R and ϵ are those in Condition 1. Therefore, if we replace Condition 1 with*

$$\delta < \frac{\pi R}{4}(\hat{r} - R) \tag{42}$$

we know that the delta chosen for any $\hat{r}$ will permit us to arrive at the conclusions of this lemma, even in cases where we have to perturb $\hat{r}$. Next we choose a sequence of $r_i > R$ converging monotonically to R for which the inequality does work. Equation (31) is still valid if we replace B_R with B_{r_i}. Equation (32) can be slightly modified using (27) to

$$\xi_{\pm}(r_i) = \sqrt{\frac{\pi R^2}{4}} \pm \sqrt{\frac{\pi R^2}{4} - 2\delta_i}. \tag{43}$$

where the $\delta_i < \delta$ and $\delta_i \to \delta$ as $i \to \infty$. Now, simply repeating the derivation in lines (36) to (39), gives

$$|B_R \cap \Sigma| = \lim_{i\to\infty} |B_{r_i} \cap \Sigma| \geq \pi R^2 - 6\delta = \lim_{i\to\infty} \pi R^2 - 6\delta_i \tag{44}$$

Now we continue with the proof of Theorem 2. Computing (again and less optimally, but sufficiently for our purposes) the change in energy when we add a ball B_r to Σ for $r \in (0, R)$, we get

$$\begin{aligned}\Delta E = E(\Sigma \cup B_r) &- E(\Sigma) &(45)\\ &\leq -H^1(\partial_*\Sigma \cap B_{r}{}^{i}_{*}) + H^1(\partial_* B_r \cap \Sigma^o_*) + \lambda|B_r \setminus \Sigma| &(46)\\ &= -\mathrm{Per}(\Sigma; B_r) + H^1(\partial_* B_r \cap \Sigma^o_*) + \lambda|B_r \setminus \Sigma|. &(47)\end{aligned}$$

By the coarea formula and properties of the measure theoretic exterior,

$$|B_r \setminus \Sigma| = |B_r \cap \Sigma^o_*| = \int_0^r H^1(\partial_* B_\xi \cap \Sigma^o_*)\, d\xi. \tag{48}$$

By the relative isoperimetric inequality applied in the ball $B_r(x_0)$,

$$\mathrm{Per}(\Sigma; B_r) \geq C \min\left\{|B_r \setminus \Sigma|^{\frac{1}{2}}, |\Sigma \cap B_r|^{\frac{1}{2}}\right\}. \tag{49}$$

Assuming

Condition 3. $6\delta < \frac{1}{4}\pi R^2$

implies $|B_R \setminus \Sigma| < \frac{1}{4}\pi R^2$. Assuming $r > \frac{R}{\sqrt{2}}$ implies that $|B_r \setminus \Sigma| \leq |\Sigma \cap B_r|$ and consequently

$$\mathrm{Per}(\Sigma; B_r) \geq C|B_r \setminus \Sigma|^{\frac{1}{2}}. \tag{50}$$

This gives a condition on ϵ:

Condition 4. $\epsilon < 1 - \frac{1}{\sqrt{2}}$.

Define $v(r) := |B_r \setminus \Sigma|$. By differentiating (48) with respect to r, and using (50) we see that the inequality concerning the change in energy given in (45) can be written as

$$E(\Sigma \cup B_r) - E(\Sigma) \leq \lambda v(r) - C\sqrt{v(r)} + v'(r). \tag{51}$$

We will use the differential expression on the right to show that the change in energy on the left has to be negative for some r close to R.

Remark 7. *Note that by choosing δ small enough, we can make $v(r)$ arbitrarily small and obtain $\lambda v(r) - C\sqrt{v(r)} < 0$; if the right hand side is positive then we have $v'(r) > 0$. This in turn means that $v(r)$ decreases as r gets smaller. We exploit this to force the right hand side to zero.*

Lemma 1. *$v'(r) - C\sqrt{v(r)} + \lambda v(r) \le 0$ for a set of $r \in ((1-\epsilon)R, R)$ with positive measure.*

Proof of lemma: Assume

$$v'(r) - C\sqrt{v(r)} + \lambda v(r) \ge 0 \text{ for a.e. } r \in ((1-\epsilon)R, R)\,, \tag{52}$$

otherwise we are done. Let $w(s) := e^{-\lambda s} v(R-s)$. Then (52) turns into

$$w'(s) + Ce^{\frac{-\lambda s}{2}} \sqrt{w(s)} \le 0 \text{ for a.e. } s \in (0, \epsilon R)\,. \tag{53}$$

with the initial condition $w(0) = |B_R \setminus \Sigma|$ and $w(s) \ge 0$. Solutions of this differential inequality can be bounded from above by solutions of the following differential equality:

$$\begin{aligned} \bar{w}' &= -Ce^{\frac{-\lambda s}{2}} \sqrt{\bar{w}}. \\ \bar{w}(0) &= |B_R \setminus \Sigma| \text{ and } \bar{w} \ge 0. \end{aligned} \tag{54}$$

The solution is

$$\begin{aligned} \sqrt{\bar{w}(s)} &= \max\left(0, \frac{C}{\lambda}\left(e^{-\frac{\lambda}{2}s} - 1\right) + \sqrt{|B_R \setminus \Sigma|}\right) \\ &= \max\left(0, \frac{CR}{2}\left(e^{-\frac{s}{R}} - 1\right) + \sqrt{|B_R \setminus \Sigma|}\right). \end{aligned}$$

Therefore if $|B_R \setminus \Sigma| \le 6\delta$ and

Condition 5. *$6\delta \le \alpha$, where α is any solution to*

$$\frac{CR}{2}\left(e^{-\frac{\epsilon R}{R}} - 1\right) + \sqrt{\alpha} = \frac{CR}{2}\left(e^{-\epsilon} - 1\right) + \sqrt{\alpha} < 0 \tag{55}$$

i.e., we have

$$\delta < \frac{C^2R^2}{24}(1 - e^{-\epsilon})^2, \tag{56}$$

then we have a set of r with positive measure in $((1-\epsilon)R, R)$ such that $v(r) = 0$ and $v'(r) = 0$. ■

This lemma immediately implies that for some $r \in ((1-\epsilon)R, R)$, $B_r \cup \Sigma$ is also a minimizer. ■

5 The Case $n > 2$

The analogs for Theorems 1 and 2 in $\mathbb{R}^n$ are:

Theorem 4. *Let Ω be a bounded, measurable subset of $\mathbf{R}^n$. Let Σ be any solution of (3). Assume that a ball B_R, $R = \frac{n}{\lambda}$ of radius R lies completely in Ω: $B_R \subset \Omega$. Then $B_R \cup \Sigma$ is also a minimizer. Moreover, if $B_R \subset \Omega^c$, then $(B_R \cup \Sigma^c)^c$ is also a minimizer.*

Theorem 5. *Given*

$$\hat{r} \in \left(R, \left(2 - \left(\frac{n-1}{n} \right)^n \right)^{\frac{1}{n}} R \right)$$

and

$$\epsilon \in \left(0, 1 - \frac{1}{2^{\frac{1}{n}}} \right)$$

we can choose $\delta > 0$ such that

$$|B_{\hat{r}} \setminus \Omega| < \delta \Rightarrow B_{(1-\epsilon)R} \cup \Sigma \text{ is also a minimizer.} \tag{57}$$

We do not present the proofs, since they are very similar to the $n = 2$ case. In particular, making the replacement $R = \frac{2}{\lambda} \to R = \frac{n}{\lambda}$ enables us to use the proof of Theorem 1, with obvious modifications, to obtain Theorem 4. Likewise, we can use the proof of Theorem 2 to prove Theorem 5, with modifications noted below.

(1) Again, $R = \frac{2}{\lambda} \to R = \frac{n}{\lambda}$,

(2) We define $\xi \equiv |B_r \cap \Sigma|^{\frac{1}{n}}$. Let α_n be the volume of the ball with unit radius in $\mathbb{R}^n$. The polynomial in (26) then gets replaced by

$$\frac{n}{R}\xi^n - n\alpha_n^{\frac{1}{n}}\xi^{n-1} + \left(\frac{2n\delta}{R} + n\alpha_n r^{n-1} - \frac{n\alpha_n r^n}{R} \right). \tag{58}$$

Since we are interested in the roots of this polynomial, we look at

$$\xi^{n-1}\left(\alpha_n^{\frac{1}{n}} R - \xi \right) = 2\delta + \alpha_n r^{n-1}(R - r). \tag{59}$$

(3) We replace the right hand side of (51) with

$$\lambda v(r) - Cv(r)^{\frac{n-1}{n}} + v'(r) \tag{60}$$

which gives us

$$w'(s) + Ce^{\frac{-\lambda s}{n}} (w(s))^{\frac{n-1}{n}} \leq 0 \tag{61}$$

in place of (53).

6 Discussion

As mentioned in the introduction, Allard [1] has recently produced an extensive study of the regularity of minimizers for a class of functionals including the L^1TV functional. In this work he uses geometric measure theory techniques originally developed to address minimal surface problems. As a result, his n cannot exceed 7. In our work we have used simpler pieces of geometric measure theory, specifically the structure theory for sets of finite perimeter.

The weaker regularity results we use – simply what one gets from Σ having finite perimeter – are not limited to $n \leq 7$.

Allard proves that the total mean curvature of minimizers is bounded by λ, as suggested by a naive calculation with the formal Euler-Lagrange equation. For spheres, this corresponds to a radius of curvature of $\frac{n-1}{\lambda}$. In our work we find that spheres (balls) of radii $\frac{n}{\lambda}$ play a critical role. Such a sphere has a total mean curvature of $\frac{n-1}{n}\lambda$. This second, bigger radius characterizes the global nature of the minimizers. Indeed if one can contain Ω in a ball of radius $\frac{n}{\lambda} - \epsilon$, where $0 < \epsilon$, then the unique solution is the empty set. This follows from a monotonicity result proved by Yin in [9]. It also follows from monotonicity results in Allard's paper [1].

Another previous work that needs to be mentioned is the work of Italo Tamanini and collaborators (see [8] and references). Instead of using knowledge of Ω to deduce properties of the minimizer Σ, they use weaker properties of Σ to establish stonger properties of the same Σ. (In particular, if a ball of a particular radius is almost contained in Σ then the ball with half the radius and same center is completely contained in Σ. This is similar to our Thereom 2.) These types of regularity properties of minimizers are not very useful for computation or in the establishment of minimizer properties based only on realistically obtainable knowledge.

There are numerous potential directions in which to advance to these results and the results reported in [1, 9]. Generalization to anisotropic energies (see [4] for example), the construction of hybrid analytic-numerical algorithms for L^1TV minimization, and the exploitation and analysis of the scale decomposition properties of the L^1TV functional are three that come easily to mind.

7 Acknowledgements

Both authors would like to acknowledge the pleasant benefit of discussions with Bill Allard. The research was made possible by funding from Los Alamos National Laboratory, the Department of Energy, and the UCLA mathematics department. The first author would also like to acknowledge the hospitality of the Institute for Pure and Applied Analysis at UCLA where he was able to work uninterrupted on the research reported here.

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